stichting
mathematisch
centrum

AFDELING NUMERIEKE WISKUNDE NW 149/83 MAART
(DEPARTMENT OF NUMERICAL MATHEMATICS)

P.J. VAN DER HOUWEN

STABILITY RESULTS FOR DISCRETE VOLTERRA EQUATIONS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65R20

Stability results for discrete Volterra equations*)

by

P.J. van der Houwen

ABSTRACT

Firstly, stability results are presented for a general class of linear multistep methods for Volterra equations, These results are obtaines by deriving a recurrence relation of finite length for the discrete Volterra equations. Secondly, the various results are illustrated by a numerical example. Finally, results of Lubich are mentioned which do not use finite recurrence relations.

KEY WORDS & PHRASES: *Volterra integral equation; numerical stability*

---

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider Volterra integral equations of the form

$$(1.1)\qquad \theta(t)y(t) = g(t) + \int_{t_0}^{t} k(t,s,y(s))ds,\quad t \in I := [t_0,T].$$

This equation is called of the *first kind* if $\theta \equiv 0$, of the *second kind* if $\theta \equiv 1$, and of the *third kind* if θ has a finite number of zeros in I. The initial or forcing function g(t) and the kernel function k(t,s,y) are prescribed, and y(t) is the unknown function.

It will be assumed that (1.1) possesses a unique solution in C [I] which is ensured if g and k are sufficiently smooth and unless $\theta \not\equiv 0$, if $k_t(t,t,y)$ is bounded away from zero for $t \in I$ and $y \in \mathbb{R}$ (for precise conditions we refer to Tricomi [26] and Anselone [3]).

In this paper we concentrate on the stability of numerical methods for solving (1.1) with fixed step size h. We will confine our considerations to a general class of linear multistep methods and to stability with respect to perturbations of the initial function g(t) on an infinite interval, i.e. $T \to \infty$. The following definition of stability will be used.

<u>DEFINITION 1.1.</u> Let y_n and y_n^* denote the numerical solutions corresponding to initial functions g and g^*, respectively, and let $g - g^* \in P[t_0,\infty]$ where $P[t_0,\infty]$ denotes a space of perturbations defined on I. Then

(a) y_n and the generating method are said to be *stable with respect to* $P[t_0,\infty]$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that

$$\max_{n\geq 0} | g(t_n) - g^*(t_n) | \leq \delta \Rightarrow \max_{n\geq 0} | y_n - y_n^* | \leq \varepsilon .$$

(b) y_n and the generating method are said to be *asymptotically stable with respect to* $P[t_0,\infty]$ if there exists a δ such that

$$\max_{n\geq 0} | g(t_n) - g^*(t_n) | \leq \delta \Rightarrow y_n - y_n^* \to 0 \quad \text{as} \quad n \to \infty . \quad \square$$

Depending on the choice of the space of perturbations $P[t_0,\infty]$ stronger and weaker forms of stability are obtained.

In deriving stability conditions so far the greater part of the papers on stability has used some (linear) test kernel for which one tries to reduce the numerical scheme to a *recurrence relation of finite length* and to which one applies the theory of linear difference equations. One class of frequently used test kernels are the *polynomial convolution kernels*

$$(1.2) \qquad K(t,s,y) = \sum_{i=0}^{m} \lambda_i (t-s)^i y.$$

Obviously, by repeated differentiation with respect to t of the corresponding

Volterra equation we can obtain a differential equation of order $m + 1$. By a similar operation of repeated differencing the numerical scheme can be often reduced to the discrete analogue of a differential equation, viz. a finite recurrence relation for y_n. Another class of suitable test kernels are the *finitely decomposable* kernels which also lead to finite recurrence relations. In Section 3 and 4 we will discuss the kernel (1.2) for $m = 1$, in Section 5 finitely decomposable kernels will be treated, and in Section 6 the various results will be compared by means of a numerical example.

Recently, Lubich [19], inspired by earlier work of Nevanlinna [22,23], has derived stability results without using finite recurrence relations. In Section 7 some of his results will be presented.

2. VLM METHODS

A simple way of discretizing the equation (1.1) consists of writing this equation with $t = t_n := t_0 + nh$ for $n = 0(1)N$, h fixed and such that $t_N := T$, and approximating the integral term by some suitably chosen quadrature rule. The numerical solution can then be obtained by solving the resulting algebraic equations successively. This method is called a *direct quadrature* (DQ) method. Such methods do not always produce satisfactory results. For instance, if Gregory quadrature rules (for a definition see e.g. [4, p. 117]) are used, equations of the first kind cannot be solved because the numerical method does not converge (see [18, 11]), and equations of the second kind in which the kernel has a large Lipschitz constant with respect to y, will often require a much smaller integration step h then necessary for representing the solution of the equation. In order to overcome these difficulties several alternative methods have been proposed (see [28, 30, 13]). These alternative methods together with the above mentioned DQ method can be described by the following *Volterra linear multistep* method (VLM method). Let $\tilde{F}_n(t)$ denote some numerical approximation to the so-called *lag term function*

$$(2.1)\qquad F(t,s) := g(t) + \int_{t_0}^{s} k(t,s,y(s))ds$$

at the point $s = t_n$ and define the *VLM formula*

$$(2.2)\qquad \sum_{i=0}^{\kappa} \alpha_i\, \theta_{n-i}\, y_{n-i} + \sum_{i=0}^{\kappa} \sum_{j=-\kappa}^{\kappa} \beta_{ij}\, \tilde{F}_{n-i}(t_{n+j}) = h \sum_{i=0}^{\kappa} \sum_{j=-\kappa}^{\kappa} \gamma_{ij}\, k(t_{n+j}, t_{n-i}, y_{n-i})$$

for $n = \kappa(1)N$, where $\theta_{n-i} := \theta(t_{n-i})$ and $y_1, \ldots y_{\kappa-1}$ are assumed to be precomputed by some starting method. Then the VLM method consists of two components: the *VLM formula* (2.2) and a *quadrature rule* for approximating $F(t,s)$. Usually, the quadrature rule is of the form

$$(2.3)\qquad \begin{aligned}\tilde{F}_n(t) &:= g(t) + h \sum_{\ell=0}^{\tilde{n}} w_{n\ell}\, k(t,t_\ell,y_\ell) \\ &= F(t,t_n) - E_n(h;t),\quad \tilde{n} = \begin{cases} \tilde{\kappa} & \text{for } n < \tilde{\kappa} \\ n & \text{for } n \geq \tilde{\kappa} \end{cases},\end{aligned}$$

where the $w_{n\ell}$ denote given quadrature weights and $\tilde{\kappa}$ is sufficiently large in order to obtain a sufficiently small approximation error $E_n(h,t)$. The lag term formula (2.3) requires the starting values $y_0, y_1, \ldots, y_{\tilde{\kappa}}$.

The parameters α_i, β_{ij} and γ_{ij} determine the accuracy and stability of the VLM method. For convergence results in the case of $\theta \equiv 0$ or $\theta \equiv 1$, we refer to [14]. Here we concentrate on the stability of VLM methods to be discussed in the remaining sections. This section is concluded with a few examples of VLM methods.

EXAMPLE 2.1. The DQ method can be presented as the simple VLM method

$$(2.4)\qquad \theta_n\, y_n - \tilde{F}_n(t_n) = 0.$$

If $E(h,t) = O(h^r)$ as $h \to 0$ uniformly for all $t_n = t_0 + nh \in I$ and the starting errors are $O(h^q)$ then the DQ method is of order $p = \min\{q-1,r\}$ if $\theta \equiv 1$ and g,k are sufficiently smooth. For $\theta = 0$ convergence is not guaranteed (see [28]). □

EXAMPLE 2.2. Consider the VLM formula

$$(2.5)\qquad 3\theta_n y_n - 4\theta_{n-1}y_{n-1} + \theta_{n-2}y_{n-2} + 3\tilde{F}_n(t_n) - 4\tilde{F}_n(t_{n+1}) + \tilde{F}_n(t_{n+2}) = \\ = 2hk(t_n,t_n,y_n)$$

which was called in [13] an *indirect backward differentiation formula* (IBD formula). It generates a method of order $p = \min\{q,r,2\}$ for second kind equations and first kind equations as well. □

3. THE BASIC TEST EQUATION

We start with the derivation of a recurrence relation of fixed length for the VLM solution of the Volterra equation

$$(3.1)\qquad \theta(t)y(t) = g(t) + \int_{t_0}^{t} f(s,y(s))ds.$$

The linear case where $f(s,y) = \xi y$ for ξ constant, is called the *basic test equation* for stability. It was proposed by Mayers [21] and extensively used by Baker and Keech [6] in deriving stability results for the DQ method.

In deriving stability results for the VLM method it is convenient to introduce the forward shift operator E and the polynomials

$$(3.2)\qquad \alpha(z) = \sum_{i=0}^{\kappa} \alpha_i z^{\kappa-i},\ \beta(z) = \sum_{i=0}^{\kappa}\Big(\sum_{j=-\kappa}^{\kappa}\beta_{ij}\Big)z^{\kappa-i},\ \gamma(t) = \sum_{i=0}^{\kappa}\Big(\sum_{j=-\kappa}^{\kappa}\gamma_{ij}\Big)z^{\kappa-i}.$$

THEOREM 3.1. *For equation* (3.1) *the* VLM *method is algebraically equivalent with the recurrence relation*

$$(3.3)\qquad \alpha(E)\theta_n y_n - h\gamma(E)f(t_n,y_n) = -E^{\kappa}\sum_{i=0}^{\kappa}\sum_{j=-\kappa}^{\kappa}\beta_{ij}g(t_{n+j}),\ n \ge 0,$$

provided that $\beta(z) \equiv 0$ (*i.e. the* VLM *is* (α,γ)*-reducible*). □

From this recurrence relation and Lemma 3.1 stated below, conclusions can be drawn on the behaviour of y_n as $n \to \infty$ (with h fixed) in the case of the basic test equation. We first state this lemma which is proved e.g. in [25, p. 205] and then give a stability result in the form of Corollary 3.1.

LEMMA 3.1. *Let* $G(z)$ *be a polynomial satisfying the root condition (that is with all its zeros on the unit disk those on the unit circle being simple zeros). Then there exists a constant* C *such that the solution of the linear, inhomogeneous difference equation* $G(E)y_n = \bar{g}_{n+m}$, $n \geq 0$ *satisfies the inequality*

$$|y_n| \leq C\left\{\max_{0\leq\ell\leq m-1}|y_\ell| + \sum_{\ell=m}^{n}|\bar{g}_\ell|\right\},\ n \geq m.$$

If $G(z)$ *is a Schur polynomial (that is all zeros are within the unit circle) then*

$$|y_n| \leq C\left\{\max_{0\leq\ell\leq m-1}|y_\ell| + \max_{m\leq\ell\leq n}|\bar{g}_\ell|\right\},\ n \geq m. \quad \square$$

COROLLARY 3.1. *Let* $\theta = 0$ *or* $\theta = 1$, *let* $\beta(z) \equiv 0$ *and let* $k(t,s,y) = \xi y$. *Then the* VLM *method is stable with respect to the space of perturbations*

(a) $P[t_0,\infty] = L^1[t_0,\infty]$ *if* $\theta\alpha(z) - h\xi\gamma(z)$ *satisfies the root condition*

(b) $P[t_0,\infty] = C\,[t_0,\infty]$ *if* $\theta\alpha(z) - h\xi\gamma(z)$ *is a Schur polynomial.* □

EXAMPLE 3.1. The VLM formula (2.5) can be characterized by the polynomials

$$\alpha(z) = 3z^2 - 4z + 1,\ \beta(z) \equiv 0,\ \gamma(z) = 2z^2.$$

Thus, the corresponding VLM method is stable with respect to $C[t_0,\infty]$: (i) for all ξ if $\theta = 0$; (ii) for those ξ such that $3z^2 - 4z + 1 - 2h\xi z^2$ is a Schur polynomial if $\theta = 1$ (this polynomial is easily recognized as the characteristic polynomial of the two-step backward differentiation method which is known to be a Schur polynomial if Re $\xi < 0$). □

EXAMPLE 3.2. Consider the VLM formula (θ=0,1)

$$\theta[y_n - y_{n-1}] + \tilde{F}_{n-1}(t_{n-1}) - \tilde{F}_{n-1}(t_n) = \frac{1}{2}h[k(t_n,t_n,y_n) + k(t_{n-1},t_n,y_n)]$$

which belongs to the class of *modified multilag* (MML) formulas proposed by Wolkenfelt [28,30]. The polynomials α,β and γ are given by

$$\alpha(z) = z - 1, \ \beta(z) = 0, \ \gamma(z) = \frac{1}{2}(z+1)$$

For $\theta = 0$ we have stability w.r.t. $L^1[t_0,\infty]$ and for $\theta = 1$ w.r.t. $C[t_0,\infty]$ provided that Re $\xi < 0$. □

The above stability results do not apply to third kind Volterra equations because the recurrence relation (3.3) when applied to the basic test kernel, does not reduce to a constant coefficient recursion. We also observe that the stability conditions expressed in Corollary 3.1 do not involve any knowledge of the lag term quadrature rule. Thus an efficient lag term formula can be conbined with a stable pair $\{\alpha,\gamma\}$ to obtain an VLM method that can be easily implemented on a computer.

In analogy with ODEs one may define the *stability region* R as the set of points $h\xi \in \mathbb{C}$ where the VLM method is stable. If R contains the whole negative axis then the method is called A_0*-stable* (when applied to the basic test equation). If the whole left half-plane is contained in R then the VLM method is called *A-stable*.

In order to see whether there exist A-stable, (α,γ)-reducible VLM methods which are convergent, we should know what conditions convergence imposes on the polynomials $\{\alpha,\gamma\}$ (see [14]).

<u>THEOREM 3.2.</u> *Let* $\beta(z) \equiv 0$. *The conditions to be imposed on the polynomials* $\{\alpha,\gamma\}$ *in order to obtain a convergent* VLM *method are: (i) if* $\theta \equiv 1$ *then* $\{\alpha,\gamma\}$ *should generate a convergent* LM *method for* ODEs; *(ii) if* $\theta \equiv 0$ *then* γ *should be a Schur polynomial.* □

Since there exist A-stable, convergent LM methods for ODEs, we conclude from Corollary 3.1 and Theorem 3.2 that there exist convergent, (α,γ)-reducible VLM methods for second kind equations which are A-stable w.r.t. $C[t_0,\infty]$. For first kind equations we see that convergence implies stability w.r.t. $L'[t_0,\infty]$.

The above considerations do not apply to e.g. the DQ methods because of the condition $\beta(z) \equiv 0$. It is possible to include such non-(α,γ)-reducible VLM formulas by imposing additional conditions on the lag term formula. We will not work this out for the basic test kernel but instead we give in the next section an analysis of the *convolution test kernel* of which the basic test kernel is a special case.

Finally, we remark that the stability criteria derived for the basic test equation may be indicative for the stability of methods applied to more general kernels of the form $K(t,s)y$. In practice, one replaces ξ by $K(t,s)$ with $t_0 \le s \le t \le T$.

4. THE CONVOLUTION TEST EQUATION

It has been observed by Kershaw [17] that the use of the basic test equation "is obviously convenient, however, its true relevance to the integral equation situation does not appear to have been thoroughly examined". In order to get some insight to what extent the stability criteria derived on the basis of equation (3.1) change if we are dealing with a more general equation, several authors have considered the convolution equation [29, 8].

$$\theta y(t) = g(t) + \int_{t_0}^{t} [\xi + \eta(t-s)]f(s,y(s))ds \qquad (4.1)$$

with ξ and η constant. The linear case is called the *convolution test equation*.

We will not restrict our considerations to (α,γ)-reducible methods; in order to facilitate an elegant analysis we require the quadrature rule used for computing the lag term to be (ρ,σ)-*reducible* (cf. [20, 31]). Let the polynomials

$$\tilde{\rho}(z) := \sum_{i=0}^{\tilde{\kappa}} \tilde{a}_i \, z^{\tilde{\kappa}-i}, \quad \sigma(z) := \sum_{i=0}^{\tilde{\kappa}} \tilde{b}_i z^i \qquad (4.2)$$

define a convergent LM method for ODEs, then the quadrature rule (2.3) is called $(\tilde{\rho},\tilde{\sigma})$-reducible if

$$\sum_{i=0}^{\tilde{\kappa}} \tilde{a}_i w_{n-i,\ell} = \begin{cases} 0 & \text{for } \ell=0(1)n-\tilde{\kappa}-1 \\ \tilde{b}_{n-\ell} & \text{for } n-\tilde{\kappa}(1)n \end{cases}, \quad n = \tilde{\kappa}, \tilde{\kappa}+1, \ldots . \qquad (4.3)$$

(We have added the tilde in order to indicate the relation with the lag term $\tilde{F}_n(t)$.) For the analysis of more general lag term formulas we refer to [6, 28,].

In addition to the polynomials $\alpha,\gamma,\tilde{\rho}$ and $\tilde{\sigma}$ we define the polynomials

$$\overline{\beta}(z) := \sum_{i=0}^{\kappa} \overline{\beta}_i z^{\kappa-i}, \overline{\beta}_i := \sum_{j=-\kappa}^{\kappa} j\,\beta_{ij}; \ \beta^{\#}(z) := \kappa\beta(z) - z\beta'(z) \qquad (4.4)$$

and similarly the polynomials $\gamma(z)$, $\gamma^{\#}(z)$, $\tilde{\rho}^{\#}(z)$ and $\tilde{\sigma}^{\#}(z)$.

The analogue of Theorem 3.1 now reads (cf. [15, 9]):

<u>THEOREM 4.1.</u> *For equations* (4.1) *the* VLM *method with* $(\tilde{\rho},\tilde{\sigma})$*-reducible lag term is algebraicly equivalent with the recurrence relation*

$$(4.5)\qquad \tilde{\rho}^{r}(E)\alpha(E)\theta_n y_n + \xi h\, \tilde{\rho}^{r-1}(E)[\beta(E)\tilde{\sigma}(E) - \gamma(E)\tilde{\rho}(E)]f(t_n,y_n)$$
$$+ \eta h^2\Big\{\tilde{\rho}^{r-1}(E)[\bar{\beta}(E)\tilde{\sigma}(E) - \bar{\gamma}(E)\tilde{\rho}(E) - \gamma^{\#}(E)\tilde{\rho}(E)]$$
$$+ \rho(E)[\beta(E)\tilde{\sigma}^{\#}(E) - \beta^{\#}(E)\tilde{\sigma}(E)] - \tilde{\rho}^{\#}(E)\tilde{\sigma}(E)\beta(E)\Big\}f(t_n,y_n)$$
$$= -\tilde{\rho}^{r}(E)E^{\kappa}\Big(\sum_{i=0}^{\kappa}\sum_{j=\kappa}^{\kappa}\beta_{ij}\, g(t_{n+j})\Big),\quad n \ge 0,$$

where $r = 1$ if $\beta(z) = 0$ and $r = 2$ otherwise. □

COROLLARY 4.1. *Let* $\theta = 0$ *or* $\theta = 1$ *and let* $k(t,s,y) = [\xi + \eta(t-s)]y$. *Then the* VLM *method with* $(\tilde{\rho},\tilde{\sigma})$*-reducible lag term is stable with respect to*

(a) $P[t_0,\infty] = L^1[t_0,\infty]$ *if*

$$(4.6)\qquad \theta\tilde{\rho}^{r}(z)\alpha(z) + \xi h\,\tilde{\rho}^{r-1}(z)[\beta(z)\tilde{\sigma}(z) - \gamma(z)\tilde{\rho}(z)]$$
$$+ \eta h^2\Big\{\tilde{\rho}^{r-1}(z)[\bar{\beta}(z)\tilde{\sigma}(z) - \gamma(z)\tilde{\rho}(z) - \gamma^{\#}(z)\tilde{\rho}(z)]$$
$$+ \tilde{\rho}(z)[\beta(z)\tilde{\sigma}^{\#}(z) - \beta^{\#}(z)\tilde{\sigma}(z)] - \tilde{\rho}^{\#}(z)\tilde{\sigma}(z)\beta(z)\Big\}$$

satisfies the root condition.

(b) $P[t_0,\infty] = C[t_0,\infty]$ *if* (4.6) *is a Schur polynomial.* □

EXAMPLE 4.1. Consider the DQ method applied to the basic test equation (i.e. η=0). Then $\alpha(z) = 1$, $\beta(z) = -1$ and $\gamma(z) = 0$ so that (4.6) reduces to

$$(4.6')\qquad \theta\,\tilde{\rho}(z) - \xi h\,\tilde{\sigma}(z).$$

For $\theta = 1$ this leads to the same stability regions which apply to the LM method $\{\tilde{\rho},\tilde{\sigma}\}$. In particular, if $\{\tilde{\rho},\tilde{\sigma}\}$ is A_0- or A-stable then the DQ method is also A_0- or A-stable (w.r.t. $C[t_0,\infty]$). For $\theta = 0$ we find that at least $\tilde{\sigma}(z)$ should satisfy the root condition. Thus, the higher order Gregory rules which are based on the Adams-Moulton methods do not generate a stable DQ method for first kind equations because $\tilde{\sigma}(z)$ do not have all its roots on the unit disc. □

In actual application, the Gregory rules are popular because of their easy implementation on a computer. The preceding example, however, shows that for second kind equations ($\theta=1$) the DQ methods have the rather modest stability regions possessed by the Adams-Moulton methods and for first kind equations the higher order methods are even unstable. This observation was precisely the reason for introducing alternative methods such as the IBD methods (cf. Example 2.2) and the MML methods (cf. Example 3.2).

As for the basic test equation one may define for the convolution test equation the stability region R which contains all points $(\xi h, \eta h^2)$ for which the VLM method is stable. The method is called *V_0-stable* if R contains the points $\{(\xi,\eta) \mid \xi < 0, \eta \le 0\}$ (cf. [8, 29]). Evidently, V_0-stability is the analogue of A_0-stability defined in the preceding section. It has already been observed that A_0-stable DQ methods do exist. This raises the question whether V_0-stable DQ methods exist. Wolkenfelt [29] proved the following negative result.

THEOREM 4.2. *For $\theta \equiv 1$ DQ methods with $(\tilde{\rho},\tilde{\sigma})$-reducible lag term cannot be V_0-stable.*

Amini [1] considered the V_0-stability for the class of *MML-formulas* defined by

(4.7)
$$\theta\alpha_0 y_n + \sum_{i=1}^{k} \alpha_i[\theta y_{n-i} + \tilde{F}_{n-i}(t_n) - \tilde{F}_{n-i}(t_{n-i})] = h \sum_{i=0}^{k} \gamma_i k(t_n, t_{n-i}, y_{n-i}).$$

THEOREM 4.3. *For $\theta \equiv 1$ MML methods with $(\tilde{\rho},\tilde{\sigma})$-reducible lag term cannot be V_0-stable.* □

Nevertheless, the MML methods behave much more stable than DQ methods (cf. [30]).

Next, we consider the class of *indirect linear multistep* (ILM) methods, an example of which has already been given in Example 2.2 . These methods are defined by the ILM formula [15]

$$(4.8)\qquad \sum_{i=0}^{k} [\theta\alpha_i y_{n-i} + \sum_{j=-i}^{k-i} \gamma_i \delta_{i+j} \tilde{F}_{n-i}(t_{n+j})] = h \sum_{i=0}^{k} \gamma_i k(t_{n-i}, t_{n-i}, y_{n-i}),$$

where $\{\delta_\ell\}$ define a numerical differentiation formula. The corresponding polynomial (4.3) is given by

$$(4.9)\qquad \tilde{\rho}(z)[\theta\alpha(z) - \xi h\,\gamma(z)] - \eta h^2 \tilde{\sigma}(z)\,\gamma(z).$$

For $\theta \equiv 1$ this polynomial is identical to the characteristic polynomial Brunner and Lambert [7] derived for their stability test equation for integro-differential equations. Since in that paper stability regions are given which do contain the points $\{(h\xi, h^2\eta) \mid \xi < 0,\ \eta \leq 0\}$, we may conclude that *there exists V_0-stable ILM methods for the second kind test equation.*

<u>EXAMPLE 4.2.</u> Let $\{\tilde{\rho},\tilde{\sigma}\}$ and $\{\alpha,\gamma\}$ correspond to the trapezoidal rule and the backward Euler rule. Then the four different methods which can be formed are all V_0-stable for the convolution test equation of the second kind. □

So far we have considered the case $\theta \equiv 1$. Next, consider the case $\theta = 0$. For the DQ method the polynomial (4.6) is given by

$$(4.10)\qquad \tilde{\rho}(z)[\theta\tilde{\rho}(z) - \xi h\,\tilde{\sigma}(z)] - \eta h^2 z[\tilde{\sigma}(z)\tilde{\rho}'(z) - \tilde{\rho}(z)\tilde{\sigma}'(z)],$$

which for $\theta = 0$ can be written in the form

$$(4.11)\qquad \tilde{\rho}(z)\tilde{\sigma}(z) - \left(-\frac{\eta}{\xi}\,h\right) z[\tilde{\sigma}(z)\tilde{\rho}'(z) - \tilde{\rho}(z)\tilde{\sigma}'(z)].$$

This polynomial can be interpreted as the characteristic polynomial of an LM method $\{\rho_1,\sigma_1\}$ for the ODE $y'(t) = (-\eta/\xi)y(t)$. If the DQ method is V_0-stable, then $-\eta/\xi$ assumes values in the range $(-\infty,0)$. Hence, we only have V_0-stability if the LM method $\{\rho_1,\sigma_1\}$ is A_0-stable. Since $\rho_1(z)$ is of degree $2\tilde{\kappa}$ and $\sigma_1(z)$ of degree $2\tilde{\kappa} - 1$ the LM method $\{\rho_1,\sigma_1\}$ cannot be V_0-stable. Thus,

<u>THEOREM 4.4.</u> *For $\theta \equiv 0$ DQ methods with $(\tilde{\rho},\tilde{\sigma})$-reducible lag term cannot be V_0-stable.*

For the MML formula (4.7) the polynomial (4.6) reduces to

$$\tilde{\rho}(z)[\theta\alpha(z) - \xi h\, \tilde{\rho}(z)] - \eta h^2\{\alpha(z)\tilde{\sigma}(z) - \alpha_0 z^{\kappa}\, \tilde{\sigma}(z) + \tilde{\rho}(z)[\kappa\gamma(z) - z\,\gamma'(z)]\}, \tag{4.12}$$

which again can be associated to an LM method $\{\rho_1,\sigma_1\}$ for the ODE $y'(t) = (-\eta/\xi)y(t)$ if $\theta = 0$. It has not yet been investigated whether this LM method can be made A_0-stable (implying V_0-stability for the MML method) by appropriate choice of the polynomials α, γ, $\tilde{\rho}$ and $\tilde{\sigma}$, and taking into account the convergence conditions.

Finally, we consider the ILM formula (4.8) with characteristic polynomial (4.9) which for $\theta = 0$ assumes the form

$$\gamma(z)[\tilde{\rho}(z) - \left(-\frac{\eta}{\xi}h\right)\tilde{\sigma}(z)]. \tag{4.13}$$

<u>THEOREM 4.5.</u> *Let the LM method $\{\tilde{\rho},\tilde{\sigma}\}$ be A_0-stable. Then for $\theta = 0$, ILM methods with $\{\tilde{\rho},\tilde{\sigma}\}$-reducible lag term are V_0-stable with respect to*

(a) $P[t_0,\infty] = L^1[t_0,\infty]$ *if $\gamma(z)$ satisfies the root condition.*

(b) $P[t_0,\infty] = C[t_0,\infty]$ *if $\gamma(z)$ is a Schur polynomial.* □

As for the basic test equation , the stability conditions based on the convolution test equation are applied to more general convolution kernels

$K(t-s)y$ by putting $\xi = K(0)$ and $\eta = K'(t)$ with $t \in I$.

5. FINITELY DECOMPOSABLE KERNELS

Instead of proceeding along the lines outlined in the preceding sections and deriving recurrence relations for the case of the polynomial convolution kernel (1.2) (cf. [2]), we approximate the kernel $k(t,s,y)$ by a *finitely decomposable* function, i.e.

$$k(t,s,y) \approx \sum_{\mu=1}^{m} g_\mu(t) f_\mu(s,y) =: \langle \vec{G}(t), \vec{F}(s,y) \rangle \tag{5.1}$$

where $\langle \, , \rangle$ denotes the inner product and $\vec{G}$, $\vec{F}$ denote vectors with components g_μ, f_μ $(\mu=1,2,\dots,m)$. If we use the approximation (5.1) then the solution $y(t)$ of (1.1) satisfies the equations

$$\begin{cases} \vec{U}'(t) = \vec{F}(t,y(t)), \; \vec{U}(t_0) = \vec{0} \\ \theta(t)y(t) = g(t) + \langle \vec{G}(t), \vec{U}(t) \rangle \end{cases} \tag{5.2}$$

The VLM method when applied to a Volterra equation with kernel of finitely decomposable form turns out to be a discretization of the system (5.2). In the following theorem which provides this relation we use the notation $\vec{F}_n := \vec{F}(t_n,y_n)$, $\vec{G}_n := \vec{G}(t_n)$.

<u>THEOREM 5.1.</u> *For kernels of finitely decomposable form the VLM method is algebraically equivalent with the recurrence relation*

$$\begin{cases} \tilde{\rho}(E)\vec{U}_n = h\tilde{\sigma}(E)\, \vec{F}_n, \; n \geq 0 \\ \sum_{i=0}^{\kappa} \alpha_i \theta_{n-i} y_{n-i} = \sum_{i=0}^{\kappa} \sum_{j=-\kappa}^{\kappa} \Big[\gamma_{ij} h \langle \vec{G}_{n+j}, \vec{F}_{n-i} \rangle \\ \qquad - \beta_{ij}\Big(g(t_{n+j}) + \langle \vec{G}_{n+j}, \vec{U}_{n-i} \rangle\Big)\Big], \; n \geq 0 \end{cases} \tag{5.3}$$

where $\{\tilde{\rho},\tilde{\sigma}\}$ *defines the lag term quadrature rule.* □

Unlike the recurrence relations presented in the preceding sections, the recurrence relation provided by this theorem generally does not have *constant coefficients* when applied to kernel functions with $f_\mu(s,y) = \xi_\mu y$. Nevertheless, it provides some insight into the stability of the VLM method as we will see in the following subsections.

It should also be observed that stability results obtained for decomposable kernels $\in C$ hold for arbitrary kernels $\in C$ because (by the Stone-Weierstrass theorem) the class of continuous decomposable kernels is dense in the class of all continuous kernels and because the VLM solution depends continuously on the kernel provided that k is sufficiently smooth [9].

5.1. Relation with ODE methods if $\theta \equiv 1$

Suppose that all coefficients in the VLM formula vanish except for $\alpha_0 = 1$, and $\beta_{00} = -1$, to obtain the *DQ method* (see Example 2.1). If $\theta \equiv 1$, then (5.3) is recognized as an LM discretization of the system (5.2). Consequently, if the LM method $\{\tilde{\rho},\tilde{\sigma}\}$ is suitable for the integration of (5.2), then the DQ method based on $\{\tilde{\rho},\tilde{\sigma}\}$ is suitable for the integration of the Volterra equation with kernel (5.1). An advantage of this approach is that the well-developed theory for ODEs can be exploited. On the other hand, one should know something about the decomposition approximating the given kernel. For a further discussion we refer to [9].

Next differentiate the second equation in (5.2) to obtain (for $\theta \equiv 1$)

$$(5.2) \qquad \begin{cases} \vec{U}'(t) = \vec{F}(t,y(t)) \\ y'(t) = g'(t) + \langle \vec{G}'(t), \vec{U}(t) \rangle + \langle \vec{G}(t), \vec{F}(t,y(t)) \rangle \end{cases}.$$

Let these differential equations be integrated by the LM methods $\{\tilde{\rho},\tilde{\sigma}\}$ and $\{\tilde{\alpha},\tilde{\gamma}\}$, respectively and replace the derivatives g' and G' by numerical approximations of the form:

$$g'(t_n) = h^{-1}\tau(E)g(t_n), \tag{5.4}$$

where $\tau(z)$ is a polynomial generating the numerical differentiation formula. The numerical scheme takes the form

$$\begin{cases} \rho(E)\vec{U}_n = h\tilde{\sigma}(E)\vec{F}(t_n,y_n), \\ \alpha(E)y_n = h\gamma(E)[h^{-1}\tau(E)g(t_n) + \langle h^{-1}\tau(E)\vec{G}(t_n),\ \vec{U}(t_n)\rangle \\ \qquad\qquad + \langle \vec{G}(t_n),\ \vec{F}(t_n,y_n)\rangle]. \end{cases} \tag{5.5}$$

A comparison with (5.3) reveals that (5.5) is a special case of a VLM formula. In [15] this type of formula was called an *indirect linear* multistep formula (see also Section 4). The stability properties of ILM formulas are largely determined by the polynomials $\{\tilde{\rho},\tilde{\sigma}\}$ and $\{\alpha,\gamma\}$, and can be chosen appropriately by using ODE stability theory.

5.2. Convolution kernels

In this section we derive a general stability result for convolution kernels of the linear form:

$$k(t,s,y) = K(t-s)y. \tag{5.6}$$

Let us first assume that k is decomposable, i.e.

$$K(t-s) = \langle \vec{G}(t),\ \vec{F}(s)\rangle. \tag{5.7}$$

Introducing the vectors

$$\vec{v}_n := [y_n,\ \vec{U}_n^T]^T,\ \vec{w}_n = [-\sum_{i=0}^{\kappa}\sum_{j=-\kappa}^{\kappa}\beta_{ij}\,g(t_{n+j}),\ 0,\ \ldots,\ 0]^T \tag{5.8}$$

the recursion (5.3) can be written in the form

$$(5.9) \qquad \sum_{i=0}^{\tilde{\kappa}} \mathcal{B}_i(n)\vec{v}_{n-i} = \vec{w}_n, \quad \kappa^* = \max\{\kappa,\tilde{\kappa}\}$$

where the matrices $\mathcal{B}_i(n)$ are given by

$$\mathcal{B}_i(n) = \begin{pmatrix} L_i & M^{(1)} & \dots & M_i^{(m)} \\ N_i^{(1)} & \tilde{a}_i I & & \\ \vdots & & \ddots & 0 \\ \vdots & 0 & & \\ N_i^{(m)} & & & \tilde{a}_i I \end{pmatrix}$$

with

$$L_i := \theta\alpha_i - h \sum_{j=-\kappa}^{\kappa} \gamma_{ij} K((j-i)h)$$

$$M_i^{(\mu)} := \sum_{j=-\kappa}^{\kappa} \beta_{ij}\, g_\mu(t_{n+j}), \quad N_i^{(\mu)} = -\tilde{b}_i h f_\mu(t_{n-i})$$

and with the convention that $L_i = M_i^{(\mu)} = 0$ for $i > \kappa$ and $\tilde{a}_i = \tilde{b}_i = 0$ for $i > \tilde{\kappa}$.

In analogy to the linear stability analysis used in ODEs one may introduce the concept of *local stability at a point* $t_{\bar{n}}$, that is we require the relation

$$(5.9') \qquad \sum_{i=0}^{\kappa^*} \mathcal{B}_i(\bar{n})\vec{v}_{n-i} = \vec{w}_n, \quad \bar{n} \text{ fixed}$$

to be stable, rather than (5.9). It is to be expected that local stability in a sequence of points $t_n, t_{n+1}, \dots, t_{n+r}$ implies "global stability" in the range $[t_n, t_{n+r}]$ provided that the matrices $\mathcal{B}_i(n)$ are slowly varying. Following [12] Theorem 5.2 can be proved.

<u>THEOREM 5.2.</u> *Let* $\theta \equiv 0$ *or* $\theta \equiv 1$ *and let* $k(t,s,y) = K(t-s)y$ *with* $K \in C[t_0,\infty]$. The *VLM method with* $(\tilde{\rho},\tilde{\sigma})$*-reducible lag term is locally stable at all points*

t_n, $n \geq k^*$ *with respect to the space of perturbations* $L^1[t_0,\infty]$ *if the polynomial*

(5.10) $$\sum_{i=0}^{\kappa^*}\sum_{j=0}^{\kappa^*} \theta\alpha_i\tilde{a}_j + h\sum_{\ell=-\kappa}^{\kappa}[\beta_{i\ell}\tilde{b}_j - \gamma_{j\ell}\tilde{a}_i]K((\ell+j)h)]z^{2\kappa^*-i-j}$$

is a Schur polynomial. Here $\tilde{a}_j = \tilde{b}_j = 0$ *for* $j > \tilde{\kappa}$ *and* $\alpha_i = \beta_{i\ell} = \gamma_{i\ell} = 0$ *for* $i > \kappa$. □

We observe that the characteristic polynomial (5.10) does not depend on n so that the local stability conditions to be derived from this theorem hold in the whole integration interval. Notice also that (5.10) only contains the function K(t) and does not refer to a particular decomposition of the form (5.7). Thus, the theorem applies to arbitrary continuous, linear convolution kernels.

In practical applications Theorem 5.2 yields complicated (local) stability conditions unless $\kappa + \kappa^*$ is small (for a worked-out example see Section 5.3). However, some insight into the local stability behaviour can be obtained if $K((\ell+j)h)$ is sufficiently close approximated by a truncated Taylor expansion.

(5.11) $$K((\ell+j)h) = \xi + (\ell+j)h\,\eta + \dots$$

where $\xi := K(0)$, $\eta = K'(0)$, ... If only one term is used we obtain on substitution into (5.10) the polynomial (4.6') derived for the basic test equation, and if two Taylor terms are used we obtain the polynomial

(5.12) $$\theta\tilde{\rho}(z)\alpha(z) + \xi h[\beta(z)\tilde{\sigma}(z) - \gamma(z)\tilde{\rho}(z)] + \eta h^2[\bar{\beta}(z)\tilde{\sigma}(z) - \gamma(z)\tilde{\rho}(z) - \bar{\gamma}^{\#}(z)\tilde{\rho}(z) + \beta(z)\tilde{\sigma}^{\#}(z)].$$

A comparison with (4.6) reveals that (5.12) and (4.6) are identical if the

VLM formula is (α,γ)-reducible $(\beta(z)\equiv 0)$. For $\beta(z) \not\equiv 0$ the characteristic polynomials differ which may be explained by observing that (5.12) characterizes the *local* stability behaviour whereas (4.6) characterizes the *global* stability of the method. A further consequence of the local stability approach is the approximation (5.11) to be valid only in a small neighbourhood of $t = 0$, whereas global stability requires the approximation to be valid in all points of the domain of definition. *Thus, adopting the validity of local stability analysis, and assuming that* $K(t)$ *and* $K'(t)$ *are slowly varying in the interval* $[0,(\kappa+\kappa^*)h]$, *we expect stability w.r.t.* $L^1[t_0,\infty]$ *if* (5.12) *(with* $\xi=K(0)$, $\eta=K'(0)$) *is a Schur polynomial.*

<u>EXAMPLE 5.1.</u> In the case of the conventional DQ method the polynomial (5.10) reduces to

$$(5.10') \qquad \theta\tilde{\rho}(z) - h \sum_{j=0}^{\tilde{\kappa}} \tilde{b}_j K(jh) z^{\tilde{\kappa}-j}$$

a result already obtained in [12]. In particular, if all coefficients $\tilde{b}_j$ but one vanish (so-called *local differentiation methods* [16]), we obtain a polynomial in which only one $K(jh)$ value is involved. For instance, if $\{\tilde{\rho},\tilde{\sigma}\}$ corresponds to a backward differentiation formula we obtain $\theta\,\rho(z) - \tilde{b}_0\,\xi h z^{\tilde{\kappa}}$ where $\xi = K(0)$. For the convolution test equation (4.1) this results in a *locally* V_0*-stable DQ method* with respect to perturbations in $L^1[t_0,\infty]$ both for $\theta = 0$ and $\theta = 1$. □

6. NUMERICAL ILLUSTRATION

We derive the various stability conditions for the DQ method generated by the *trapezoidal rule* when applied to the second kind equation (cf. Garey [10])

$$(6.1)\qquad y(t) = \frac{1}{2}\lambda(1-t^2)\ell n(1+t) + \frac{3}{4}\lambda t^2 - (\tfrac{1}{2}\lambda+1)t + 1 - \lambda\int_0^t \ell n(1+t-s)y(s)ds$$

with exact solution $y(t) = 1 + t$.

When the stability conditions based on the *basic test equation* are applied, we find from (4.6'), with $\theta \equiv 1$, $\tilde{\rho} = z - 1$, $\tilde{\sigma} = \frac{1}{2}(z+1)$ and $\xi = K(t,s)$, the stability condition (w.r.t. $L^1[0,\infty]$):

(6.2) $\quad z - 1 - \frac{1}{2}hK(t,s)(z+1),\ 0 \le s \le t \le \infty$ *should satisfy the root condition.*

Evidently, this condition is satisfied in the case of equation (6.1) for all $h\lambda \ge 0$.

Using the convolution test equation, we find from (4.10) the stability condition (w.r.t. $L^1[0,\infty]$):

(6.3) $\quad (z-1)^2 - \frac{1}{2}hK(0)(z^2-1) - h^2zK'(t),\ t \in [t_0,\infty]$ *should satisfy the root condition.*

Applying Hurwitz criterion this condition reduces to

$$(6.3')\qquad K(0) \le 0;\ K'(t) < 0,\ t \in [0,\infty];\ h < \frac{2}{\sqrt{|K'(t)|}},\ t \in [0,\infty].$$

For equation (6.1) this leads us to the condition $h < 2\sqrt{(1+t)/\lambda}$, $\lambda > 0$.

Next we consider the polynomial (5.12) yielding the local stability condition (w.r.t. $L^1[0,\infty]$)

(6.4) $\quad z - 1 - \frac{1}{2}hK(0)(z+1) - \frac{1}{2}h^2K'(0)$ *should be a Schur polynomial.*

This results in

$$(6.4')\qquad K(0) \le 0;\ K'(0) < 0;\ h < \frac{2}{\sqrt{|K'(0)|}},$$

so that in case of equation (6.1) the local condition $h < 2/\sqrt{\lambda}$, $\lambda > 0$ is obtained.

Finally, we choose the polynomial (5.10') as our starting point to obtain the "rigorous" local condition (again w.r.t. $L^1[0,\infty]$)

(6.5) $\quad [1 - \frac{1}{2}hK(0)]z - [1 + \frac{1}{2}hK(h)]$ *should be a Schur polinomial,*

leading to the condition

(6.5') $\quad [K(h) + K(0)][4 + hK(h) - hK(0)] < 0,$

and in the case of (6.1) to the step size condition $h < 4/\lambda \ell n(1+h)$, $\lambda > 0$.

Summarizing, the following stability conditions are found for (6.1):

test equations used		condition	($\lambda = 100$, $T = 4$)
basic test equation		$\lambda h \geq 0$	no condition
convolution test equation		$h < 2\sqrt{(1+t)/\lambda}$	$h < .44$
general convol. eq.	appr. (5.11)	$h < 2/\sqrt{\lambda}$	$h < .20$
	rigorous	$h < 4/\lambda \ell n(1+h)$	$h < .21$

In order to test these results we have integrated (6.1) with $\lambda = 100$ and $T = 4$ to obtain the accuracies (measured by the number of correct significant digits $sd := -{}^{10}\log|\text{relative error}|$) listed in the following table:

h	.24	.23	.22	.21	.20	.19	.18	.17
sd	-7.81	-6.47	-4.87	-2.77	1.88	2.70	2.61	2.65

These figures clearly show for this example the reliability of the local stability conditions and the too optimistic prediction if the kernel is approximated by the basic or convolution test kernels.

7. NEGATIVE DEFINITE CONVOLUTION KERNELS

Recently Lubich [19] has developed global stability results for $(\tilde{\rho},\tilde{\sigma})$-reducible DQ methods when applied to second kind equations with convolution kernels of the form

(7.1) $\qquad k(t,s,y) = \xi\, K(t-s), \quad \operatorname{Re} \xi < 0$

where K(t) is a continuous, *positive definite* function $\in L^1[\mathbb{R}_+]$. Here, a continuous function $a : \mathbb{R} \to \mathbb{C}$ is said to be positive definite if

$$\sum_{i,j} a(t_i - t_j) z_i \bar{z}_j \geq 0$$

for any choice of finite sequences $\{t_i\}$ and $\{z_i\}$ with $t_i \in \mathbb{R}$ and $z_i \in \mathbb{C}$. Similarly, a sequence $\{a_n\}_{-\infty}^{\infty}$ is said to be positive definite if

$$\sum_{i,j} a_{i-j} z_i \bar{z}_j \geq 0$$

for any choice of finite complex sequences $\{z_i\}$.

This work extends earlier work of Nevanlinna [22, 23]. Without proof we give the basic lemma's and the stability theorem from Lubich's paper.

LEMMA 7.1. *Let* $h > 0$. *If* $a : \mathbb{R}_+ \to \mathbb{C}$ *and* $\{\omega_\ell\}_0^\infty$ *are positive definite, then the sequence* $\{\omega_\ell a(\ell h)\}_0^\infty$ *is again positive definite.* □

LEMMA 7.2. *(Toeplitz, Carathéodory). The sequence* $\{a_n\}_0^\infty$ *is positive definite iff it is bounded and* $\operatorname{Re} \Sigma_{\ell=0}^{\infty} a_\ell z^\ell \geq 0$ *in* $|z| < 1$. □

LEMMA 7.3. *Let* $\omega(z) := \rho(z^{-1})/\sigma(z^{-1}) = \Sigma_{\ell=0}^{\infty} \omega_\ell z^\ell$. *The stability region* R *of the LM method* $\{\rho, \sigma\}$ *contains an open disc (stability disc* $\mathcal{D}$*) of radius* r *in the left half-plane touching the origin iff there exists a number* c *such that the sequence* $\{\omega_0 + c, \omega_1, \omega_2, \ldots\}$ *is a positive definite sequence. Here* $c = 1/(2r)$. □

LEMMA 7.4. *(Lubich, Paley-Wiener). Let* $\{y_n\}$ *satisfy the recurrence relation*

(7.2) $$y_n = g_n + \sum_{\ell=0}^{n} b((n-\ell)h) y_\ell, \quad n \geq 0, \; h > 0$$

where $b(t) \in L^1[\mathbb{R}_+]$ *and let*

(7.3) $$\sum_{\ell=0}^{\infty} b(\ell h) z^{\ell} \neq 0 \text{ for } |z| \leq 1.$$

(a) $y_n \to 0$ *whenever* $g_n \to 0$ *as* $n \to \infty$ *iff* (6.3) *is satisfied.*

(b) y_n *is bounded whenever* g_n *is bounded as* $n \to \infty$ *iff* (7.3) *is satisfied.* □

THEOREM 7.1. *(Lubich). Let* R *contain a stability disc* $\mathcal{D}$ *of radius* r*, let* $h\,\xi \in \mathcal{D}$*, and let* $k(t,s,y)$ *be of the definite convolution form* (7.1) *with* $K(0) = 1$. *Then*

(a) $y_n \to 0$ *whenever* $g(t_n) \to 0$ *as* $n \to \infty$.

(b) y_n *is bounded whenever* $g(t_n)$ *is bounded as* $n \to \infty$. □

Sketch of the proof. First the numerical scheme is written in the form (6.2) so that by Lemma 7.4 it remains to verify Paley-Wiener's condition (6.3) with $b(\ell h) = \xi h\, \omega_\ell K(\ell h)$. By Lemma 7.1 and 7.3 the sequence $\{(\omega_0+(2r)^{-1})K(0), \omega_1 K(h), \omega_2 K(2h), \ldots\}$ is positive definite, hence by Lemma 7.2

$$\mathrm{Re} \sum_{\ell=0}^{\infty} \omega_\ell\, K(\ell h) z^{\ell} \geq -\frac{K(0)}{2r} = -\frac{1}{2r} \quad \text{for} \quad |z| \leq 1.$$

Thus $\sum_{\ell=0}^{\infty} \omega_\ell\, K(\ell h) z^{\ell} \neq 1/(\xi h)$ for $\xi h \in \mathcal{D}$ which is just the Paley-Wiener condition. □

REFERENCES

[1] AMINI, S., *Private communication, paper in preparation.*

[2] AMINI, S., C.T.H. BAKER, P.J. van der HOUWEN & P.H.M. WOLKENFELT, *Stability analysis of numerical methods for Volterra integral equations with polynomial convolution kernels*, J. of Integral Egns. 5, -.-, 1983.

[3] ANSELONE, P.M. (ed)., *Nonlinear integral equations*, University of Wisconsin Press, Madison, 1964.

[4] BAKER, C.T.H., *The numerical treatment of integral equations*, Clarendon Press, oxford, 1977.

[5] BAKER, C.T.H., *Structure of recurrence relations in the study of stability in the numerical treatment of Volterra integral and integro-differential equations*, J. of Integral Eqns. 2, 11-29, 1980.

[6] BAKER, C.T.H. & M.S. KEECH, *Stability regions in the numerical treatment of Volterra integral equations*, SIAM J. Numer. Anal. 15, 394-417,1978

[7] BRUNNER, H. & J.D. LAMBERT, *Stability of numerical methods for Volterra integro-differential equations*, Computing 12, 75-89, 1974.

[8] BRUNNER, H., S.P. NØRSETT & P.H.M. WOLKENFELT, *On V_0-stability of numerical methods for Volterra integral equations of the second kind*, Report NW 84/80, Mathematisch Centrum, Amsterdam, 1980.

[9] BRUNNER, H. & P.J. van der HOUWEN, *The numerical solution of Volterra equations*, North-Holland, Amsterdam, to appear in 1984.

[10] GAREY, L., *Solving nonlinear second kind Volterra equations by modified increment methods*, SIAM J. Numer. Anal. 12, 501-508, 1975.

[11] GLADWIN, C.J. & R. JELTSCH, *Stability of quadrature rule methods for first kind Volterra integral equations*, BIT 14, 144-151, 1974.

[12] HOUWEN, P.J. van der & P.H.M. WOLKENFELT, *On the stability of multistep formulas for Volterra integral equations of the second kind*, Computing 24, 341-347, 1980.

[13] HOUWEN, P.J. van der & H.J.J. te RIELE, *Backward differentiation type formulas for Volterra integral equations of the second kind*, Numer. Math. 37. 205-217, 1981.

[14] - , *Linear multistep methods for Volterra integral and integro-differential equations*, in preparation.

[15] - , In: *Treatment of integral equations by numerical methods* (eds. C.T.H. Baker and G.F. Miller), Academic Press, 1982.

[16] KEECH, M.S., *A third order, semi-explicit method in the numerical solution of first kind Volterra integral equations*, BIT 17, 312-320, 1977.

[17] KERSHAW, D., *Volterra equations of the second kind*, In: Numerical solution of integral equations (eds. L.M. Delves & J. Walsh), Clarendon Press, Oxford, 1974.

[18] LINZ, P., *Numerical methods for Volterra integral equations of the first kind*, Computer J. 12, 393-397, 1969.

[19] LUBICH, Ch., *On the stability of linear multistep methods for Volterra equations*, Preprint nr. 190, Sonderforschungsbereich 123, Heidelberg, 1982.

[20] MATTHYS, J., *A-stable linear multistep methods for Volterra integro-diffrential equations*, Numer. Math. 27, 85-94, 1976.

[21] MAYERS, D.F., In: *Numerical solution of ordinary and partial differential equations* (ed. L. Fox), Pergamon, 1962.

[22] NEVANLINNA, O., *Positive quadratures for Volterra equations*, Computing 16, 349-357, 1976.

[23] - , *On the numerical solution of some Volterra equations on infinite intervals*, Report Inst. Mittag-Leffler, Djursholm, 1976.

[24] PALEY, R.E.A.C. & N. WIENER, *Fourier transforms in the complex domain*, Amer. Math. Soc. Providence, R.I., 1934.

[25] STETTER, H.J., *Analysis of discretization methods for ordinary differential equations*, Springer Verlag, Berlin, 1973.

[26] TRICOMI, F.G., *Integral equations*, Interscience, 1957.

[27] WOLKENFELT, P.H.M., P.J. van der HOUWEN & C.T.H. BAKER, *Analysis of numerical methods for second kind Volterra equations by imbedding techniques*, J. Integral Eq. 3, 61-82, 1981.

[28] WOLKENFELT, P.H.M., *The numerical analysis of reducible quadrature methods for Volterra integral and integro-differential equations*, Thesis, Mathematical Centrum, 1981.

[29] - , *Stability analysis of reducible quadrature methods for Volterra integral equations of the second kind*, ZAMM 61, 499-501, 1981.

[30] - , *Modified multilag methods for Volterra functional equations*, to appear in Math. Comp. 1983.

[31] - , *The construction of reducible quadrature rules for Volterra integral and integro-differential equations*, IMA J. Numer. Anal. 2, 131-152, 1982.